Unit I

Differential Equations of First Order and First Degree

INTRODUCTION

An equation of the form $F\left(x, y, \frac{dy}{dx}\right) = 0$ in which x is the independent variable and $\frac{dy}{dx}$ appears with first degree is called a first order and first degree differential equation. It can also be written in the form $\frac{dy}{dx} = f(x, y)$ or in the form $Mdx + Ndy = 0$, where M and N are functions of x and y. Generally, it is difficult to solve the first order differential equations and in some cases they may not possess any solution. There are certain standard types of first order, first degree equations. In this chapter we shall discuss the methods of solving them.

VARIABLES SEPARABLE

If the equation is of the form $f_1(x)\,dx = f_2(y)\,dy$ then, its solution, by integration is $\int f_1(x)dx = \int f_2(y)\,dy + C$

where C is an arbitrary constant.

Example 1. Solve $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$

Solution. The given equation is

$$\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$$

or $$e^y\,dy = (e^x + x^2)\,dx$$

Integrating, $e^y = e^x + \frac{x^3}{3} + c$, where C is an arbitrary constant is the required solution.

Example 2. Solve $\left(y - x\frac{dy}{dx}\right) = a\left(y^2 + \frac{dy}{dx}\right)$

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Solution. The given equation is

$$y - x\frac{dy}{dx} = ay^2 + a\frac{dy}{dx}$$

or $$(a + x)\frac{dy}{dx} = (y - ay^2)$$

or $$\frac{dy}{y - ay^2} = \frac{dx}{a + x}$$

or $$\frac{dy}{y - ay^2} = \frac{dx}{a + x}$$

or $$\frac{dy}{y(1 - ay)} = \frac{dx}{x + a}$$

or $$\left[\frac{a}{1 - ay} + \frac{1}{y}\right]dy = \frac{dx}{x + a},$$ resolving into partial fractions.

Integrating, $[-\log(1 - ay) + \log y] = \log(x + a) + \log C$

where C is an arbitrary constant

or $$\log\left(\frac{y}{1 - ay}\right) = \log\{C(x + a)\}$$

or $$\frac{y}{1 - ay} = C(x + a)$$

or $y = C(x + a)(1 - ay)$ is the required solution.

Example 2. Solve $(x + y)^2 \frac{dy}{dx} = a^2$

Solution. Let $x + y = v$

then from (1) on differentiating with respect to x, we have

$$1 + \frac{dy}{dx} = \frac{dv}{dx}$$

or $$\frac{dy}{dx} = \left(\frac{dv}{dx} - 1\right)$$

Substituting these values from (1) and (2) in the given equation, we get

$$v^2\left[\frac{dv}{dx}-1\right]=a^2$$

or $$v^2\,\frac{dv}{dx}=a^2+v^2$$

or $$\frac{v^2}{a^2+v^2}\,dv=dx$$

or $$\frac{a^2+v^2-a^2}{a^2+v^2}\,dv=dx$$

or $$\left[1-\frac{a}{a^2+v^2}\right]dv=dx$$

Integrating,

$$v-a^2\,\frac{1}{a}\,\tan^{-1}\left(\frac{v}{a}\right)=x+c$$

where c is an arbitrary constant

or $$v-a\tan^{-1}(v/a)=x+c$$

or $$(x+y)-a\tan\{(x+y)/a\}=x+c,\ \text{from (1)}$$

or $$y-a\tan^{-1}\{(x+y)/a\}=C$$

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Linear Differential Equations

A differential equation of the form $\frac{dy}{dx} + Py = Q$

where P and Q are constants or functions of x alone (and not of y) is called a linear differential equation of the first order in y

its integrating factor $= e^{\int P\,dx}$

Multiplying both sides of (1) by this integrating factor (I.F.) and then integrating we get

$y.\,e^{\int Pdx} = C + \int Q.\,e^{\int Pdx}\,dx$, where C is an arbitrary constant, is the complete solution of (1)

Example Solve $\frac{dy}{dx} + 2y\tan x = \sin x$, given that $y = 0$ when $x = \pi/3$.

Solution. Here $P = 2\tan x$ and $Q = \sin x$

$\therefore$ Integrating factor $= e^{\int Pdx} = e^{\int 2\tan x\,dx}$

$= e^{2\log\sec x}$

$= e^{\log(\sec x)^2} = \sec^2 x$

Multiplying the given equation by $\sec^2 x$, we get

$$\sec^2 x\left(\frac{dy}{dx} + 2y\tan x\right) = \sin x\sec^2 x$$

or $\frac{d}{dx}(y\sec^2 x) = \sec x\tan x$

Integrating both sides with respect to x, we get

$y\sec^2 x = C + \int \sec x\tan x\,dx$, where C is an arbitrary constant

or $\quad y\sec^2 x = C + \sec x \quad (1)$

it is given that when $x = \pi/3$, $y = 0$

$\therefore$ from (1) $0 \times \sec^2\frac{\pi}{3} = C + \sec\frac{\pi}{3}$

or $$0 = C + 2 \quad \therefore \sec\frac{\pi}{3} = 2$$

$\therefore$ from (1) the required solution is

$$y \sec^2 x = -2 + \sec x$$

or $$y = -2\cos^2 x + \cos x$$

Example . Solve $(1 + y^2)\,dx + (x - e^{\tan^{-1}y})\,dy = 0$

Solution. The given equation can be written as

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{e^{\tan^{-1}y}}{1+y^2}$$

Therefore the integrating factor $= e^{\int \frac{1}{1+y^2}dy}$

$$= e^{\tan^{-1}y}$$

Multiplying both sides of (1) by the integrating factor and integrating, we have

$$x.\, e^{\tan^{-1}y} = C + \int \frac{e^{\tan^{-1}y}}{1+y^2} \times e^{\tan^{-1}y}\,dy$$

where C is an arbitrary costant

or $$x\, e^{\tan^{-1}y} = C + \int e^{2t}\,dt, \text{ where } t = \tan^{-1}y$$

$$= C + \frac{1}{2}e^{2t}$$

or $$x.\, e^{\tan^{-1}y} = C + \frac{1}{2}e^{2\tan^{-1}y}$$

Example . Solve $\frac{dy}{dx} + y\cos x = \frac{1}{2}\sin 2x$

Solution. Here $P = \cos x$ and $Q = \frac{1}{2}\sin 2x = \sin x \cos x$

$\therefore$ Integrating factor $= e^{\int P dx} = e^{\int \cos x\, dx} = e^{\sin x}$

Multiplying the given equation by the integrating factor $e^{\sin x}$ and integrating with respect to x, we get

$y.\,e^{\sin x} = C + \int e^{\sin x} \sin x \cos x\, dx$, where C is an arbitrary constant.

or $\quad y.\,e^{\sin x} = C + \int e^{t}\, t\, dt, \quad$ where $t = \sin x$

$$= C + t.\,e^{t} - e^{t}$$

$$= C + e^{\sin x}(\sin x - 1)$$

or $\quad y.\,e^{\sin x} = C + e^{\sin x}(\sin x - 1)$

Linear Differential Equations with Constant Coefficients and Applications

INTRODUCTION

A differential equation is of the form

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + a_2 \frac{d^{n-2}y}{dx^{n-2}} + \dots\dots\dots + a_n y = Q \qquad (1)$$

where $a_1, a_2, \dots\dots a_n$ are constants and Q is a function of x only, is called a linear differential equation of n^{th} order. Such equations are most important in the study of electro-mechanical vibrations and other engineering problems.

The operator $\frac{d}{dx}$ is denoted by D.

$\therefore\ D^n y + a_1 D^{n-1} y + \dots\dots\dots\dots + a_n y = Q$

or $\quad f(D)\, y = Q$

where $f(D) = D^n + a_1 D^{n-1} + \dots\dots\dots\dots + a_n$

Solution of the Differential Equation

If the given equation is

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + a_2 \frac{d^{n-2}y}{dx^{n-2}} + \dots\dots\dots + a_n y = 0 \qquad (1)$$

or $\quad (D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots\dots\dots + a_n)\, y = 0 \qquad (2)$

Let $y = e^{mx}$

$\Rightarrow \quad D^r y = m^r e^{mx},\ 1 \le r \le n$

$\therefore$ Then from equation (2)

$m^n + a_1 m^{n-1} + a_2 m^{n-2} + \dots\dots\dots + a_n)\, e^{mx} = 0$

$y = e^{mx}$ is a solution of (1), if

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$m^n + a_1 m^{n-1} + a_2 m^{n-2} + \ldots\ldots\ldots\ldots + a_n = 0$

This equation is called the auxiliary equation.

Case 1. When Auxiliary Equation has Distinct and real Roots

Let $m_1, m_2, \ldots\ldots\ldots\ldots m_n$ are distinct roots of the auxiliary equation, then the general solution of (1) is $y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \ldots\ldots\ldots\ldots + C_n e^{m_n x}$

where $C_1, C_2, \ldots\ldots\ldots C_n$ are arbitrary constants

Illustration. Solve the differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 54y = 0$$

Solution. The given equation is

$$(D^2 + 3D - 54)y = 0$$

Here auxiliary equation is

$$m^2 + 3m - 54 = 0$$

or $\quad (m + 9)(m - 6) = 0$

$\Rightarrow \quad m = 6, -9$

Hence the general solution of the given differential equation is $y = C_1 e^{6x} + C_2 e^{-9x}$

Case II. When Auxiliary Equation has real and some equal roots.

If the auxiliary equation has two roots equal, say $m_1 = m_2$ and others are distinct say $m_3, m_4, \ldots\ldots\ldots m_n$. In this Case the general solution of the equation (1) is

$y = (C_1 + C_2 x) e^{m_1 x} + C_3 e^{m_3 x} + \ldots\ldots\ldots + C_n e^{m_n x}$

where $C_1, C_2, C_3 \ldots\ldots\ldots C_n$ are arbitrary constants

Illustration. Solve the differential equation

$$(D^4 - D^3 - 9D^2 - 11D - 4)y = 0$$

Solution. The auxiliary equation of the give equation is

$m^4 - m^3 - 9m^2 - 11m - 4 = 0$

or $\quad (m + 1)^3 (m - 4) = 0$

$\Rightarrow \quad m = -1, -1, -1, 4$

Hence, the required solution is

$$y = (C_1 + C_2 x + C_3 x^2) e^{-x} + C_4 e^{4x}$$

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or $\quad y = (C_1 x^2 + C_2 x + C_3) e^{-x} + C_4 e^{4x}$

Case III. When the auxiliary equation has imaginary roots

If there are one pair of imaginary roots say $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$ i.e. $\alpha \pm i\beta$ say then the required solution is

$e^{x\,(\text{Real part})}$ [C_1 {cos (imaginary part) x} + C_2 {sin (imaginary part) x}]

i.e. $\quad e^{\alpha x} [C_1 \cos \beta x + C_2 \sin \beta x]$

or $\quad y = C_1 e^{\alpha x} \cos (\beta x + C_2)$

Illustration. Solve $(D^2 - 2D + 5) y = 0$

Solution. Here the auxiliary equation is

$$m^2 - 2m + 5 = 0$$

or $$m = \frac{1}{2}\left[2 \pm \sqrt{(4-20)}\right] = 1 \pm 2i$$

∴ The required solution is

$y = e^x (C_1 \cos 2x + C_2 \sin 2x)$

Particular Integral (P.I.)

when the equation is

$$D^n + a_1 D^{n-1} + \dots\dots\dots + a_n) y = Q$$

or $$f(D) y = Q$$

The general solution of f (D) y = Q is equal to the sum of the general solution of f (D) y = 0 called complementary function (C.F.) and any particular integral of the equation f(D) y = Q

∴ General solution = C.F. + P.I.

A particular integral of the differential equation

f(D) y = Q is given by $\frac{1}{f(D)} Q$

Methods of finding Particular integral

(A)

Case I. P.I., when Q is of the form of e^{ax}, where a is any constant and $f(a) \neq 0$

we know that $\quad D(e^{ax}) = a\,e^{ax}$

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$$D^2 (e^{ax}) = a^2 e^{ax}$$

$$D^3 (e^{ax}) = a^3 e^{ax}$$

In general $$D^n (e^{ax}) = a^n e^{ax}$$

$$\therefore f(D)(e^{ax}) = f(a) e^{ax}$$

or $$\frac{1}{f(D)} f(D) e^{ax} = \frac{1}{f(D)} f(a) e^{ax}$$

or $$e^{ax} = f(a) \frac{1}{f(D)} e^{ax} \because f(a) \text{ is constant}$$

or $$\frac{1}{f(a)} e^{ax} = \frac{1}{f(D)} e^{ax}$$

Hence $\text{P.I.} = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$, if $f(a) \neq 0$

Case II. P.I., when Q is of the form of e^{ax}, and $f(a) = 0$

Then $\frac{1}{f(D)} (e^{ax}) = e^{ax} \frac{1}{f(D+a)} 1$, which shows that if e^{ax} is brought to the left from the right of $\frac{1}{f(D)}$, then D should be replaced by $(D + a)$

Another method for Exceptional Case

If $f(a) = 0$, then

$$\text{P.I.} = \frac{1}{f(D)} e^{ax} = x \frac{1}{f'(D)} e^{ax}$$

$$= x \frac{e^{ax}}{f'(a)}, \text{ if } f'(a) \neq 0$$

If $f'(a) = 0$, then

$$\text{P.I.} = \frac{1}{f(D)} e^{ax}$$

$$= x^2 \frac{e^{ax}}{f''(a)}, \text{ if } f''(a) \neq 0$$

Example 1. Solve $(D^2 - 2D + 5) y = e^{-x}$

Solution. Here auxiliary equation is $m^2 - 2m + 5 = 0$, whose roots are

$$m = -1 \pm 2i$$

$\therefore$ C.F. $= e^{-x}[C_1 \cos 2x + C_2 \sin 2x]$, where C_1 and C_2 are arbitrary constants

and $$\text{P.I.} = \frac{1}{D^2 - 2D + 5} e^{-x}$$

$$= \frac{1}{(-1)^2 - 2(-1) + 5} e^{-x} \quad \because \text{here } a = -1$$

$$= \frac{1}{8} e^{-x}$$

$\therefore$ The required solution is y = C.F. + P.I.

i.e. $y = e^{-x} (C_1 \cos 2x + C_2 \sin 2x) + \frac{1}{8} e^{-x}$

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Example 2. Solve $(D-1)^2(D^2+1)^2 y = e^x$

Solution. Here the auxiliary equation is

$$(m-1)^2(m^2+1)^2 = 0$$

or $$m = 1, 1, \pm i, \pm i$$

$\therefore$ C.F. $= (C_1 + C_2 x)e^x + (C_3 x + C_4)\cos x + (C_5 x + C_6)\sin x$

where c's are arbitrary constant

and $$\text{P.I.} = \frac{1}{(D-1)^2(D^2+1)^2} e^x$$

$$= \frac{1}{(D-1)^2} \frac{1}{(1^2+1)^2} e^x$$

$$= \frac{1}{(D-1)^2} \frac{1}{(2)^2} e^x$$

$$= \frac{1}{(D-1)^2} \frac{1}{4} e^x$$

$$= e^x \frac{1}{(D+1-1)^2} \frac{1}{4}$$

$$= e^x \frac{1}{D^2} \frac{1}{4} = \frac{1}{4} e^x \frac{1}{D^2}(1) = \frac{1}{4} e^x \frac{x^2}{2} = \frac{1}{8} x^2 e^x$$

$\therefore$ The required solution is y = C. F + P.I.

or $$y = (C_1 x + C_2)e^x + (C_3 x + C_4)\cos x + (C_5 x + C_6)\sin x + \frac{1}{8} x^2 e^x$$

(B) (i) P.I. when Q is of the form sin ax or cos ax and $f(-a^2) \neq 0$

$$\frac{1}{f(D^2)} \sin ax = \frac{1}{f(-a^2)} \sin ax, \text{ if } f(-a^2) \neq 0$$

$$\& \frac{1}{f(D^2)} \cos ax = \frac{1}{f(-a^2)} \cos ax, \text{ if } f(-a^2) \neq 0$$

(ii) P.I. when Q is of the form sin ax or cos ax and $f(-a^2) = 0$

Example . Solve $(D^2 - 5D + 6)y = \sin 3x$

Solution. Its auxiliary equation is $m^2 - 5m + 6 = 0$ which gives

$m = 2, 3$

$\therefore$ C.F. $= C_1 e^{2x} + C_2 e^{3x}$, where C_1 and C_2 are arbitrary constants and $$\text{P.I.} = \frac{1}{D^2 - 5D + 6} \sin 3x$$

$$= \frac{1}{-3^2 - 5D + 6} \sin 3x, \text{ replacing } D^2 \text{ by } -3^2$$

$$= \frac{1}{-(5D+3)} \sin 3x = \frac{-1}{(5D+3)(5D-3)} (5D-3) \sin 3x$$

$$= \frac{-1}{(25D^2 - 9)} (5D-3) \sin 3x$$

$$= \frac{-1}{\{25(-3^2) - 9\}} (5D-3) \sin 3x$$

$$= \frac{1}{234} [5D(\sin 3x) - 3\sin 3x]$$

$$= \frac{1}{234} [5 \times 3 \cos 3x - 3 \sin 3x]$$

$$= \frac{1}{78} (5 \cos 3x - \sin 3x)$$

Hence the required solution is

$$y = C_1 e^{2x} + C_2 e^{3x} + \frac{1}{78} (5 \cos 3x - \sin 3x)$$

(C) To find P.I. when Q is of the form x^m

In this case $\text{P.I} = \frac{1}{f(D)} x^m$, where m is a positive integer

To evaluate this we take common the lowest degree from f(D), so that the remaining factor reduces to the form [1 + F (D)] or [1 – F (D)]. Now take this factor in the numerator with a negative index and expand it by Binomial theorem in powers of D upto the term D^m, (Since other higher derivatives of x^m will be zero) and operate upon x^m. The following examples will illustrate the method.

Example 12. Solve $(D^3 - D^2 - 6D) y = x^2 + 1$ Where $D = \frac{d}{dx}$

Solution. Here the auxiliary equation is $m^3 - m^2 - 6m = 0$

or $m(m^2 - m - 6) = 0$

or $m(m-3)(m+2) = 0$

or $m = 0, 3, -2$

$\therefore$ $C_1 F = C_1 e^{0x} + C_2 e^{3x} + C_3 e^{-2x}$

or C.F. $= C_1 + C_2 e^{3x} + C_3 e^{-2x}$, where C_1, C_2 and C_3 are arbitrary constants.

and $$\text{P.I.} = \frac{1}{D^3 - D^2 - 6D} (x^2 + 1)$$

$$= \frac{1}{-6D\left(1 + \frac{1}{6}D - \frac{1}{6}D^2\right)} (x^2 + 1)$$

$$= -\frac{1}{6D}\left[1 - \left(\frac{D}{6} - \frac{D^2}{6}\right) + \left(\frac{D}{6} - \frac{D^2}{6}\right)^2 \cdots\cdots\cdots\right](x^2 + 1)$$

$$= -\frac{1}{6D}\left[1 + x^2 - \frac{1}{6}D(x^2) + \frac{7}{36}D^2(x^2)\right] \qquad \because D(1) = 0$$

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(E) **To show that $\frac{1}{D-a}Q = e^{ax}\int Qe^{-ax}\,dx$, where Q is a function of x.**

Proof Let $y = \frac{1}{D-a}Q$

Then $(D-a)\,y = Q$, operating both sides with $D-a$

or $\frac{dy}{dx} - ay = Q$, Which is a linear equation in y whose integrating factor is e^{-ax} its solution is $ye^{-ax} = \int Q\,e^{-ax}\,dx$, neglecting the constant of integration as P.I. is required

or $\quad y = e^{ax}\int Q\,e^{-ax}\,dx$

or $\quad \frac{1}{D-a}Q = e^{ax}\int Q\,e^{-ax}\,dx$

Example: Solve $(D^2 + a^2)\,y = \sec ax$

Solution. The auxiliary equation is $m^2 + a^2 = 0$ or $m = \pm ai$

$\therefore$ C.F. $= C_1\cos ax + C_2\sin ax$, where C_1 and C_2 are arbitrary constants

and $\text{P.I} = \frac{1}{D^2+a^2}\sec ax = \frac{1}{2ia}\left[\frac{1}{D-ia} - \frac{1}{D+ia}\right]\sec ax \qquad (1)$

Now $\quad \frac{1}{D-ia}\sec ax = e^{iax}\int \sec ax.\,e^{-iax}\,dx$

$$= e^{iax}\int \sec ax.\,(\cos ax - i\sin ax)\,dx$$

$$= e^{iax}\int (1 - i\tan ax)\,dx$$

$$= e^{iax}\left[x + \left(\frac{i}{a}\right)\log\cos ax\right]$$

$$\frac{1}{D+ia}\sec ax = e^{-iax}\int \sec ax.\,e^{iax}\,dx$$

$$= e^{-iax}\int \sec ax\,(\cos ax + i\sin ax)\,dx$$

$$= e^{-iax}\int (1 + i\tan ax)\,dx$$

$$= e^{-iax}\left[x - \left(\frac{i}{a}\right)\log\cos ax\right]$$

$\therefore$ From (1) we, have

$$\text{P.I} = \frac{1}{2ia}\left[e^{iax}\left\{x + \frac{i}{a}\log\cos ax\right\} - e^{-iax}\left\{x - \frac{i}{a}\log\cos ax\right\}\right]$$

$$= \left[x\left(\frac{e^{iax} - e^{-iax}}{2ia}\right) + \frac{i}{a}(\log\cos ax).\left(\frac{e^{iax} + e^{-iax}}{2ia}\right)\right]$$

$$= \left(\frac{x}{a}\right)\sin ax + \frac{1}{a^2}\cos ax.\,(\log\cos ax)$$

$\therefore$ The required solution is

$$y = \text{C.F.} + \text{P.I.}$$

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or $\quad y = C_1\cos ax + C_2\sin ax + \left(\frac{x}{a}\right)\sin ax + \left(\frac{1}{a^2}\right)\cos ax.\log\cos ax$

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Equations Reducible To Linear Equations with Constant Coefficients

INTRODUCTION

Now the shall study two such forms of linear differential equations with variable coefficients which can be reduced to linear differential equations with constant coefficients by suitable substituitions.

1. Cauchy's Homogeneous Linear Equations

A differential equation of the form

$$x^n \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots\ldots + P_n y = X$$

where $P_1, P_2, \ldots\ldots\ldots P_n$ are constants and X is either a function of x or a constant is called Cauchy-Euler homogeneous linear differential equation.

The solution of the above homogenous linear equation may be obtained after transforming it into linear equation with constant coefficients by using the substitution.

By the substitution $x = e^z$ or $z = \log_e x$; $\therefore \frac{dz}{dx} = \frac{1}{x}$

Now $\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$

$$\therefore x\frac{dy}{dx} = \frac{dy}{dz} \qquad (1)$$

Again $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)$

$$= \frac{x\frac{d^2y}{dz^2}\frac{dz}{dx} - \frac{dy}{dz}}{x^2} = \frac{x\frac{d^2y}{dz^2}\frac{1}{x} - \frac{dy}{dz}}{x^2}$$

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$$\therefore x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz} \qquad (2)$$

Also $$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}\left[\frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)\right]$$

$$= \frac{x^2\left[\frac{d^3y}{dz^3}\frac{dz}{dx} - \frac{d^2y}{dz^2}\frac{dz}{dx}\right] - 2x\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)}{x^4}$$

Substituting $\frac{dz}{dx} = \frac{1}{x}$ and simplifying, we get

$$x^3 \frac{d^3y}{dx^3} = \frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz} \qquad (3)$$

Using $x\frac{d}{dx} = \frac{d}{dz} = D$, in (1), (2) and (3)

we get $$x\frac{dy}{dx} = Dy$$

$$x^2 \frac{d^2y}{dx^2} = D(D-1)y$$

$$x^3 \frac{d^3y}{dx^3} = D(D-1)(D-2)y$$

In general, we have

$$x^n \frac{d^ny}{dx^n} = D(D-1)(D-2)\ldots\ldots\ldots\ldots(D-n+1)y$$

Using, these results in homogeneous linear equation, it will be transformed into a linear differential equation with constant coefficients.

Example : . Solve $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = x^2 \log x$

Solution. On changing the independent variable by substituting

$$x = e^z \text{ or } z = \log_e x \text{ and } \frac{d}{dz} = D$$

The differential equation becomes

$$[D(D-1) - D - 3]y = ze^{2z}$$

or $$(D^2 - 2D - 3)y = ze^{2Z}$$

Now the auxiliary equation is $m^2 - 2m - 3 = 0$

$\Rightarrow$ $m = 3, -1$

Hence, the C.F. $= C_1 e^{3Z} + C_2 e^{-Z} = C_1 x^3 + \frac{C_2}{x}$

and $\quad$ $$\text{P.I.} = \frac{1}{D^2 - 2D - 3} ze^{2z}$$

$$= e^{2z} \frac{1}{(D+2)^2 - 2(D+2) - 3} z = e^{2z} \frac{1}{D^2 + 4 + 4D - 2D - 4 - 3} z$$

$$= e^{2z} \frac{1}{D^2 + 2D - 3} z$$

$$= e^{2z} \frac{1}{-3\left(1 - \frac{2D}{3} - \frac{D^2}{3}\right)} z$$

$$= \frac{e^{2z}}{-3} \left[1 - \left(\frac{2D}{3} + \frac{D^2}{3}\right)\right]^{-1} z$$

$$= \frac{e^{2z}}{-3} \left[1 + \frac{2D}{3} + \frac{D^2}{3} + \ldots\ldots\right] z$$

$$= \frac{e^{2z}}{-3} \left(z + \frac{2}{3}\right) = e^{2z} \left(-\frac{z}{3} - \frac{2}{9}\right)$$

$$= x^2 \left(-\frac{1}{3} \log_e x - \frac{2}{9}\right)$$

Hence solution of the given differential equation is

$$y = C_1 x^3 + \frac{C_2}{x} + x^2 \left(-\frac{1}{3} \log_e x - \frac{2}{9}\right)$$

2. Legendre's linear differential equation (Equation reducible to homogeneous form)

An equation of the form

$$(ax+b)^n \frac{d^n y}{dx^n} + k_1 (ax+b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + k_n y = X \qquad (1)$$

Where a, b, k_1, k_2,..........k_n are all constants and X is a function of x, is called Legendre's linear equation.

Such equations can be reduced to linear equations with constant coefficients by substituting $ax + b = e^z$ i.e. $z = \log(ax+b)$

Then if $D = \frac{d}{dz}$, $\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{a}{ax+b}\frac{dy}{dz}$

i.e. $(ax+b)\frac{dy}{dx} = aDy$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{a}{ax+b}\frac{dy}{dz}\right) = \frac{-a^2}{(ax+b)^2}\frac{dy}{dz} + \frac{a}{ax+b}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx}$$

$$= \frac{a^2}{(ax+b)^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

i.e. $(ax + b)^2 \frac{d^2y}{dx^2} = a^2 D(D-1)y$

Similarly $(ax + b)^3 \frac{d^3y}{dx^3} = a^3 D(D-1)(D-2)y$ and so on.

After making these replacements in (1), there results a linear equation with constant coefficients.

Example Solve $(1 + x)^2 \frac{d^2y}{dx^2} + (1 + x)\frac{dy}{dx} + y = 4 \cos \log (1 + x)$

Solution. put $1 + x = e^z$ and $\frac{d}{dz} = D$

Hence the given differential equation becomes

$[D(D-1) + D + 1]\, y = 4 \cos z$

$\therefore$ Auxiliary equation is

$$D^2 + 1 = 0 \text{ or } D = \pm i$$

$\therefore$ C.F. $= C_1 \cos (z + C_2) = C_1 \cos [\log (1 + x) + C_2]$

and P.I $= \frac{1}{D^2 + 1} 4 \cos z = 4. \frac{z}{2} \sin z$

$= 2z \sin z$

$= 2 \log (1 + x) \sin \log (1 + x)$

Hence the required solution is

$y = C_1 \cos [\log (1 + x) + C_2] + 2 \log (1 + x) \sin \log (1 + x)$

METHOD OF VARIATION OF PARAMETERS

Method of variation of parameters enables to find solution of any linear non homogeneous differential equation of second order even (with variable coefficients also) provided its complimentary function is given (known). The particular integral of the non-homogeneous equation is obtained by varying the parameters i.e. by replacing the arbitrary constants in the C.F. by variable functions.

Consider a linear non-homogeneous second order differential equation with variable coefficients

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = X(x) \qquad (1)$$

Suppose the complimentary function of (1) is $= C_1y_1(x) + C_2y_2(x)$ (2)

so that y_1 and y_2 satisfy

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = 0$$

Unit I

In method of variation of parameters the arbitrary constants C_1 and C_2 in (2) are replaced by two unknown functions u (x) and v (x).

Let us assume particular integral is $= u(x)\, y_1(x) + v(x)\, y_2(x)$ (3)

where $$u = \int \frac{-X y_2}{y_1 y_2' - y_1' y_2}\, dx$$

and $$v = \int \frac{X y_1}{y_1 y_2' - y_1' y_2}\, dx$$

on putting the values of u and v in (3) we get P.I

Thus, required general solution = C.F + P.I

Example Apply the method of variation of parameters to solve $\frac{d^2y}{dx^2} + y = \tan x$

Solution. The auxiliary equation is $m^2 + 1 = 0 \Rightarrow m = \pm i$

$\therefore$ C.F. $= C_1 \cos x + C_2 \sin x$ (1)

Here $y_1 = \cos x$, $y_2 = \sin x$

Therefore $y_1 y_2' - y_1' y_2 = \cos^2 x + \sin^2 x = 1$

Let us suppose P.I $= u.y_1 + v\, y_2$ (2)

where

$$u = \int \frac{-X y_2}{y_1 y_2' - y_1' y_2}\, dx = -\int \frac{\sin x \tan x}{1}\, dx$$

$$= -\int \frac{\sin^2 x}{\cos x}\, dx = -\int \frac{1 - \cos^2 x}{\cos x}\, dx$$

$$= \int (\cos x - \sec x)\, dx$$

$$= \sin x - \log(\sec x + \tan x)$$

& $$v = \int \frac{X y_1}{y_1 y_2' - y_1' y_2}\, dx = \int \frac{\tan x \cos x}{1}\, dx$$

$$= \int \sin x\, dx = -\cos x$$

Putting the values of u and v in (2), we get

$$\text{P.I} = u\, y_1 + v\, y_2$$

$$= [\sin x - \log(\sec x + \tan x)] \cos x - \cos x \sin x$$

$$= -\cos x \log(\sec x + \tan x)$$

Therefore, complete solution is

$$y = C_1 \cos x + C_2 \sin x - \cos x \log(\sec x + \tan x)$$

Unit I

SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS (Solution by Changing dependent and independent variables)

INTRODUCTION

The general form of linear differential equation of the second order may be written as

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \qquad (1)$$

where P, Q and R are functions of x only. There is no general method for the solution of this type of equations. Some particular methods used to solve these equations are, change of independent variables, Variation of parameters and removal of first order derivatives etc. As this kind of differential equations are of great significance in physics, especially in connection with vibrations in mechanics and theory of electric circuit. In addition many profound and beautiful ideas in pure mathematics have grown out to the study of these equations.

Method I: Complete solution is terms of known integral belonging to the complementary function (i.e. part of C.F. is known or one solution is known).

Let u be a part of complementary function of equation (1) and v is remaining solution of differential equation (1)

Then the complete solution of equation (1) is

$$y = uv \qquad (2)$$

$$\Rightarrow \qquad \frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx} \text{ and } \frac{d^2y}{dx^2} = v\frac{d^2u}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + u\frac{d^2v}{dx^2}$$

Putting these values in equation (1) then, we get

$$v\frac{d^2u}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + u\frac{d^2v}{dx^2} + P\left(v\frac{du}{dx} + u\frac{dv}{dx}\right) + Quv = R$$

or $$v\left[\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right] + u\left[\frac{d^2v}{dx^2} + P\frac{dv}{dx}\right] + 2\frac{du}{dx}\frac{dv}{dx} = R \qquad (3)$$

Since u is a part of C.F. i.e. solution of (1)

$$\therefore \qquad \frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu = 0$$

Unit I

Hence equation (3) becomes

$$u\left(\frac{d^2v}{dx^2} + P\frac{dv}{dx}\right) + 2\frac{du}{dx}\frac{dv}{dx} = R$$

or $$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u} \qquad (4)$$

Let $$\frac{dv}{dx} = z, \text{ so that } \frac{d^2v}{dx^2} = \frac{dz}{dx}$$

Equation (4) becomes

$$\frac{dz}{dx} + \left(P + \frac{2}{u}\frac{du}{dx}\right)z = \frac{R}{u},$$

which is linear in z, Hence z can be determined

We obtain v, by integration the relation $\frac{dv}{dx} = z$

$$\Rightarrow \qquad v = \int z\,dx + C_1$$

Therefore, the solution of (1) is $y = u\left[\int z\,dx + C_1\right]$

i.e. $\qquad y = uv$

Remark. Solving by the above method, u determined by inspection of the following rules

(1) If $P + Qx = 0$, then $u = x$

(2) If $1 + P + Q = 0$, then $u = e^x$

(3) If $1 - P + Q = 0$, then $u = e^{-x}$

(4) If $1 + \frac{P}{a} + \frac{Q}{a^2} = 0$, then $u = e^{ax}$

(5) If $2 + 2Px + Qx^2 = 0$, then $u = x^2$

(6) If $m(m-1) + Pmx + Qx^2 = 0$, then $u = x^m$

Example Solve $y'' - 4xy' + (4x^2 - 2)y = 0$ given that $y = e^{x^2}$ is an integral induced in the complementary function.

Solution. The given equation may be written as

$$\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 2)y = 0$$

Here $P = -4x$, $Q = 4x^2 - 2$, $R = 0$

and $u = e^{x^2}$, so that $\frac{du}{dx} = 2xe^{x^2}$

Let $y = uv \Rightarrow y = e^{x^2}v \qquad (1)$

we know that

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = 0, \text{ As } R = 0$$

$$\Rightarrow \quad \frac{d^2v}{dx^2} + \left(\frac{2}{e^{x^2}} 2xe^{x^2} - 4x\right)\frac{dv}{dx} = 0$$

$$\Rightarrow \quad \frac{d^2v}{dx^2} = 0$$

$$\Rightarrow \quad \frac{dv}{dx} = C$$

$$\Rightarrow \quad v = C_1x + C_2$$

Hence the complete solution is $y = e^{x^2} v$

or $\quad y = e^{x^2}(C_1x + C_2)$

Method II. Normal form (Removal of first derivative)

Let $\quad \frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \qquad (1)$

putting $y = uv$, we get

$$v\left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right) + u\left(\frac{d^2v}{dx^2} + P\frac{dv}{dx}\right) + 2\frac{du}{dx}\frac{dv}{dx} = R$$

$$\Rightarrow \quad \frac{d^2v}{dx^2} + \left(\frac{2}{u}\frac{du}{dx} + P\right)\frac{dv}{dx} + v\left(\frac{1}{u}\frac{d^2u}{dx^2} + \frac{P}{u}\frac{du}{dx} + Q\right) = \frac{R}{u} \qquad (2)$$

But the first order derivative must be remove

so $\quad \frac{2}{u}\frac{du}{dx} + P = 0 \Rightarrow \quad \frac{du}{u} = -\frac{1}{2}Pdx$

$$\Rightarrow \quad \log u = -\int\frac{P}{2}dx$$

$$\Rightarrow \quad u = e^{-\int\frac{P}{2}dx}$$

Since $\quad \frac{du}{dx} = -\frac{Pu}{2} \quad \Rightarrow \quad \frac{d^2u}{dx^2} = -\frac{1}{2}\left[P\frac{du}{dx} + u\frac{dP}{dx}\right]$

$$\Rightarrow \quad \frac{d^2u}{dx^2} = -\frac{1}{2}\left[P\left(-\frac{Pu}{2}\right) + u\frac{dP}{dx}\right] = \frac{P^2u}{4} - \frac{u}{2}\frac{dp}{dx}$$

From (2) $\frac{d^2v}{dx^2} + v\left(\frac{P^2}{4} - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{2} + Q\right) = \frac{R}{u}$

$$\Rightarrow \quad \frac{d^2v}{dx^2} + v\left(Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}\right) = \frac{R}{u}$$

$$\Rightarrow \quad \frac{d^2v}{dx^2} + Iv = \frac{R}{u} \qquad (3)$$

This equation is called normal form of equation (1)

where $\quad I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}$

Unit I

Example : . Solve

$$\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 1)y = -3e^{x^2}\sin 2x$$

Solution. Here $P = -4x$, $Q = 4x^2 - 1$, $R = -3e^{x^2}\sin 2x$

so $$I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2 = 4x^2 - 1 - \frac{1}{2}(-4) - \frac{1}{4}(-4x)^2$$

$$= 4x^2 - 1 + 2 - 4x^2 = 1$$

$$u = e^{-\frac{1}{2}\int P dx} = e^{-\frac{1}{2}\int(-4x)\,dx}$$

$$= e^{2\int x\,dx} = e^{x^2}$$

Then substituting these values in the equation

$$\frac{d^2v}{dx^2} + Iv = \frac{R}{u}, \text{ We have}$$

$$\frac{d^2v}{dx^2} + v = \frac{-3e^{x^2}\sin 2x}{e^{x^2}} = -3\sin 2x$$

its C.F $= C_1\cos x + C_2\sin x$

and $$P.I = -\frac{1}{D^2+1}2\sin 2x = -3\frac{1}{-2^2+1}\sin 2x$$

$$= \sin 2x$$

Thus $v = C_1\cos x + C_2\sin x + \sin 2x$

Therefore required solution is $y = uv$

or $y = e^{x^2}(C_1\cos x + C_2\sin x + \sin 2x)$

Method III. Change of independent variable

consider $$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \qquad (1)$$

Let us change the independent variable x to z and $z = f(x)$.

Then $$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} \qquad (2)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dz}\frac{dz}{dx}\right) = \frac{dy}{dz}\frac{d^2z}{dx^2} + \left(\frac{dz}{dx}\right)^2\frac{d^2y}{dx^2} \qquad (3)$$

Putting the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1) we get

$$\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\frac{d^2z}{dx^2} + P\frac{dy}{dz}\frac{dz}{dx} + Qy = R$$

or $$\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \left(P\frac{dz}{dx} + \frac{d^2z}{dx^2}\right)\frac{dy}{dz} + Qy = R$$

or
$$\frac{d^2y}{dz^2} + \frac{\left(P\frac{dz}{dx} + \frac{d^2z}{dx^2}\right)}{\left(\frac{dz}{dx}\right)^2}\frac{dy}{dz} + \frac{Q}{\left(\frac{dz}{dx}\right)^2}y = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

$$\Rightarrow \quad \frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1 \qquad (4)$$

where
$$P_1 = \frac{\left(P\frac{dz}{dx} + \frac{d^2z}{dx^2}\right)}{\left(\frac{dz}{dx}\right)^2},\ Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} \text{ and } R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

Equation (4) is solved either by taking $P_1 = 0$ or Q_1 = a constant

Example Solve by changing the independent variable

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} + 4x^3y = x^5$$

Solution. Given equation is

$$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} + 4x^2y = x^4 \qquad (1)$$

Here $P = -\frac{1}{x}, Q = 4x^2$ and $R = x^4$

On changing the independent variable x to z, the equation (1) transformed as

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1y = R_1 \qquad (2)$$

where
$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{4x^2}{\left(\frac{dz}{dx}\right)^2} = \text{constant} = 1 \text{ say}$$

or
$$\left(\frac{dz}{dx}\right)^2 = 4x^2$$

$$\Rightarrow \quad \frac{dz}{dx} = 2x$$

$$\Rightarrow \quad z = x^2$$

$$\Rightarrow \quad \frac{d^2z}{dx^2} = 2$$

$$P_1 = \frac{P\frac{dz}{dx} + \frac{d^2z}{dx^2}}{\left(\frac{dz}{dx}\right)^2} = \frac{\left\{2 + \left(\frac{-1}{x}\right)2x\right\}}{4x^2} = 0$$

$$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{x^4}{4x^2} = \frac{x^2}{4} = \frac{z}{4}$$

Unit I

on putting the values of P_1, Q_1 and R_1 in (2), we get

$$\frac{d^2y}{dz^2} + y = \frac{z}{4}$$

or $$(D^2 + 1)\, y = \frac{z}{4}$$

its A.E. is $m^2 + 1 = 0 \quad \Rightarrow \quad m = \pm i$

$\therefore$ $\quad C.F = C_1 \cos z + C_2 \sin z$

or $\quad C.F = C_1 \cos x^2 + C_2 \sin x^2$

and $$P.I = \frac{1}{D^2+1}\,\frac{z}{4} = \frac{1}{4}(1 + D^2)^{-1}\, z$$

$$= \frac{1}{4}(1 - D^2 + \ldots\ldots\ldots)\, z$$

$$= \frac{z}{4}$$

$$= \frac{x^2}{4}$$

Hence the complete solution is y = C.F + P.I

or $$y = C_1 \cos x^2 + C_2 \sin x^2 + \frac{x^2}{4}$$

SIMULTANEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS:

In Several applied mathematics problems, there are more than one dependent variables, each of which is a function of one independent variable, usually say time **t**. The formulation of such problems leads to a system of simultaneous linear differential equation with constant coefficients. Such a system can be solved by the method of elimination. Laplace transform method, using matrices and short cut operator methods.

Example Solve $\frac{dx}{dt} + y = \sin t,\ \frac{dy}{dt} + x = \cos t \quad x(o) = 2,\ y(o) = 0$

Solution. We have

$$\frac{dx}{dt} + y = \sin t \qquad (1)$$

$$\frac{dy}{dt} + x = \cos t \qquad (2)$$

Differentiating (1) w.r.t. 't' we have

$$\frac{d^2x}{dt^2} + \frac{dy}{dt} = \cos t \qquad (3)$$

Using (2) in (3) we get

$$\frac{d^2x}{dt^2} - x = 0 \quad \Rightarrow \quad (D^2 - 1)x = 0$$

its auxiliary equation is

$m^2 - 1 = 0 \quad \Rightarrow \quad m = \pm 1$

$\therefore \qquad x = C_1 e^t + C_2 e^{-t}$ (4)

$\Rightarrow \qquad \frac{dx}{dt} = C_1 e^t - C_2 e^{-t}$

putting this value of $\frac{dx}{dt}$ in (1) we get

$y = \sin t - C_1 e^t + C_2 e^{-t}$ (5)

Using given conditions

$$\left.\begin{array}{l}\text{from (iv) } C_1 + C_2 = 2 \\ \text{from (v) } -C_1 + C_2 = 0\end{array}\right\} \Rightarrow C_1 = C_2 = 1$$

putting these values of C_1 and C_2 in (4) & (5) we get

$$x = e^t + e^{-t}$$

and $\qquad y = \sin t - e^t + e^{-t}$

is the required solution

On changing the independent variable x to z, the equation (1) transformed as

$$\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1 \qquad (2)$$

where $\qquad Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{4x^2}{\left(\frac{dz}{dx}\right)^2} = \text{constant} = 1 \text{ say}$

where $\qquad Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{4x^2}{\left(\frac{dz}{dx}\right)^2} = \text{constant} = 1 \text{ say}$

or $\qquad \left(\frac{dz}{dx}\right)^2 = 4x^2$

$\Rightarrow \qquad \frac{dz}{dx} = 2x \Rightarrow z = x^2 \Rightarrow \frac{d^2z}{dx^2} = 2$

$$P_1 = \frac{P\frac{dz}{dx} + \frac{d^2z}{dx^2}}{\left(\frac{dz}{dx}\right)^2} = \frac{\left\{2 + \left(\frac{-1}{x}\right)2x\right\}}{4x^2} = 0 \quad R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{x^4}{4x^2} = \frac{x^2}{4} = \frac{z}{4}$$

on putting the values of P_1, Q_1 and R_1 in (2), we get

$$\frac{d^2y}{dz^2} + y = \frac{z}{4}$$

or $\qquad (D^2 + 1)y = \frac{z}{4}$